

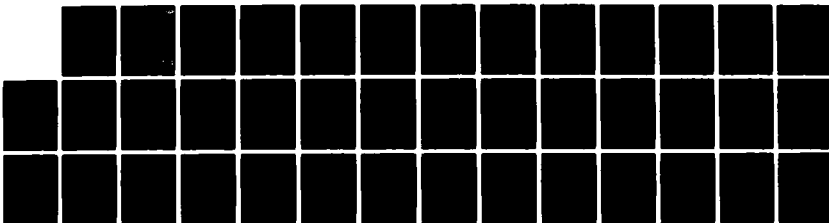
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IDENTIFICATION OF A CLASS OF FILTERED POISSON PROCESSES 1//
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STATISTICS D DE BRUCQ ET AL. 1981 N00014-75-C-0491

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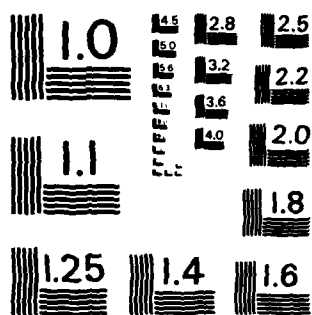
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TITLE : IDENTIFICATION OF A CLASS OF FILTERED POISSON PROCESSES

Authors : DE BRUCQ Denis* - GUALTIEROTTI Antonio**

Summary : The formulation of a model is an essential step in the experimental process of understanding a phenomenon. In this paper a new class of filtered Poisson processes is introduced : the amplitude has a law which is spherically invariant and the filter is real, linear and causal. It is shown how such a model can be identified from experimental data.

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Research supported by ONR contracts

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. ADA135371	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Identification of a Class of Filtered Poisson Processes		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Denis De Brucq & Antonio Gualtierotti		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0491 N00014-81-K-0373
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics University of North Carolina Chapel Hill, North Carolina		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 042-269 SRO 105
11. CONTROLLING OFFICE NAME AND ADDRESS Statistics & Probability Program Office of Naval Research Arlington, VA 22217		12. REPORT DATE
		13. NUMBER OF PAGES 39 39
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Filtered Poisson processes; spherically invariant processes; wave propagation		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The formulation of a model is an essential step in the experimental process of understanding a phenomenon. In this paper a new class of filtered Poisson processes is introduced: the amplitude has a law which is spherically invariant and the filter is real, linear and causal. It is shown how such a model can be identified from experimental data.		

INTRODUCTION

The process introduced in this paper serves as a useful model for the study of random wave propagation problems and certain areas of electronics and biochemistry. An initial phenomenon, modeled as a Poisson point process whose effect is described by a spherically invariant random variable C is observed through a causal linear filter whose impulse response is G . The observed process has the form

$$Z(\omega, t) = \sum_{j=-\infty}^{\infty} C_j(\omega) G(t - \tau_j(\omega)), \quad -\infty < t < +\infty$$

where τ_j is the time of the j the jump of the original Poisson process.

Spherically invariant laws are mixtures of normal ones and have previously served in the study of wave propagation problems

The first section of the paper contains a characterization of such laws based on Choquet's representation theorem [4] and some examples (the exponential and the student probability laws).

In the second part of the paper one finds the characteristic function of the process Z defined above, which is strongly stationary. This is used in the third part to show that the odd moments of Z vanish and to compute the moments of order two and four.

The identification of the process Z requires that the impulse response G be estimated knowing the moments of Z . This can be done if the filter is assumed to have minimal phase, a concept frequently used in automation (cf [6]).

The fourth part of the paper presents an algorithm for approximating G based on the Fourier transform.

The final and fifth part of the paper is devoted to the estimation of the parameter λ of the original Poisson process and of the mixing law μ which yields the spherically invariant one used in the model. Here again estimation is by the method of moments : the estimator of λ is obtained on the solution of a linear equation and to estimate μ one assumes it is a convex combination of point masses and adjusts the classical solution of the moment problem to the case of a support contained in the positive half-line.

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We wish thank Professor C. R. BAKER of the University
of North Carolina at Chapel Hill (Department of Statistics)
for many helpful discussions.

I - CHARACTERIZATION AND EXAMPLES OF SPHERICALLY INVARIANT RANDOM VARIABLES

The problem considered here is that of determining a class of spherically invariant distributions (see [5]) on the real line. The extreme points of this set are the Gaussian laws P_σ with density :

$$(1) \quad \frac{dP_\sigma}{dx}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

where dx is the Lebesgue measure and one wants to characterize the closed convex set generated by the family $\{P_\sigma, \sigma > 0\}$. In terms of characteristic functions, one wants to solve the equation :

$$(2) \quad \phi(u) = \int_0^\infty e^{-\frac{a^2 u^2}{2}} \mu(da) = \int_{-\infty}^{+\infty} e^{iux} P(dx)$$

where ϕ is the characteristic function of a spherically invariant distribution P for μ if ϕ is given and for ϕ , if μ is.

I.1 - Lemma :

If P is spherically invariant*, P is symmetric with respect to the origin.

Proof : If P is spherically invariant, then by definition or reference

$$\phi(u) = \int_{-\infty}^{+\infty} e^{iux} P(dx) = \int_0^\infty e^{-\frac{a^2 u^2}{2}} \mu(da)$$

Since $\phi(u) = \phi(-u)$, P is invariant under the transformation $x \rightarrow -x$. ■

* Some author takes a larger class than the one considered here

We know that $\phi(0) = 1$ (then $\mu(\mathbb{R}_+) = 1$) and that ϕ is continuous in a neighborhood of zero.

1.2 - Lemma :

If P is spherically invariant, there exists ψ in $C^\infty([0, \infty[)$ (with derivatives of any order in $]0, \infty[$) such that for every real u

$$(3) \quad \phi(u) = \psi(u^2)$$

Proof : Define ψ by the relation

$$\psi(v) \triangleq \int_{\mathbb{R}_+} e^{-\frac{a^2}{2} v} d\mu(a)$$

Then for every positive v and natural integer n

$$\int_{\mathbb{R}_+} \left| -\frac{a^2}{2} \right|^n e^{-\frac{a^2}{2} v} d\mu(a) < \infty$$

so that ψ may be differentiated arbitrarily often in order to obtain :

$$\frac{d^n \psi(v)}{dv^n} = \int_{\mathbb{R}_+} \left(-\frac{a^2}{2} \right)^n e^{-\frac{a^2}{2} v} d\mu(a)$$

The transformation defined by $T : a \mapsto b \triangleq \frac{a^2}{2}$ is a bijection of \mathbb{R}_+ . Let $\nu = \mu \circ T^{-1}$, the image ν of μ by T .

$$\text{Then } \psi(v) = \int_{\mathbb{R}_+} e^{-b v} d\nu(b)$$

1.3 - Theorem : ([4] p. 237) :

There exists a positive measure ν on \mathbb{R}_+ such that

$$\forall v > 0 \quad \psi(v) = \int_{\mathbb{R}_+} e^{-b v} d\nu(b)$$

if and only if

ψ is real valued on \mathbb{R}_+ , ψ is \mathcal{C}^∞ and $\forall n \geq 0 \quad (-1)^n \frac{d^n \psi}{dv^n} > 0$

1.4 - Theorem :

P is spherically invariant on $(\mathbb{R}, \mathcal{R})$ if and only if

(4) P is symmetric with respect to the origin

(5) the characteristic function ϕ of P belongs to $\mathcal{C}^\infty(\mathbb{R} \setminus \{0\})$

and the function ψ , defined by

(6) $\forall u \in \mathbb{R} \quad \phi(u) \triangleq \psi(u^2)$ is such that

$$\forall n \in \mathbb{N} \quad \forall v > 0 \quad (-1)^n \frac{d^n \psi}{dv^n}(v) \geq 0$$

Proof : If (4), (5) and (6) hold, one has, by 1-3, that

$$(7) \quad \psi(v) = \int_{\mathbb{R}_+} e^{-b \sqrt{v}} dv(b)$$

This can be written as

$$\phi(u) = \psi(u^2) = \int_{\mathbb{R}_+} e^{-\frac{a^2 u^2}{2}} d\mu(a) \text{ which, as we know is}$$

the characteristic function of a spherically invariant probability.

If P is spherically invariant, (4), (5) and (6) follow from lemmas 1 and 2. ■

1.5 - Example :

The double exponential $\frac{\lambda}{2} e^{-\lambda|x|}$ is spherically invariant.

Indeed a straightforward calculation yields :

$$\int_{-\infty}^{+\infty} e^{iux} \frac{\lambda}{2} e^{-\lambda|x|} dx = \frac{\lambda^2}{\lambda^2 + u^2} = \int_0^{\infty} e^{-\frac{a^2 u^2}{2}} e^{-\frac{a^2 \lambda^2}{2}} \lambda^2 a da$$

The result is obtained by setting

$$\frac{d\mu}{da}(a) = \lambda^2 a e^{-\frac{a^2 \lambda^2}{2}} \quad \text{or}$$

$$\psi(v) = \frac{\lambda^2}{\lambda^2 + v} \quad \text{and}$$

$$\frac{d^n}{dv^n} \psi(v) = (-1)^n \frac{\lambda^2}{(\lambda^2 + v)^{n+1}}$$

1.6 - Example :

$$\phi(u) = e^{\lambda(e^{-\alpha u^2} - 1)} \quad \lambda > 0 \quad \alpha > 0 \text{ is the characteristic}$$

function of a spherically invariant function.

Define T by $a \rightarrow \frac{a^2}{2\alpha}$ and ν by $\nu = \mu \circ T^{-1}$.

Since T is a bijection of \mathbb{R}_+

$$\phi(u) = \int_0^{\infty} e^{-\frac{a^2 u^2}{2}} \mu(da) = \int_0^{\infty} e^{-b\alpha u^2} \nu(db)$$

Choose for ν a Poisson law with parameter λ . Then

$$\phi(u) = \int_0^{\infty} e^{-\frac{a^2 u^2}{2}} \mu(da) = e^{\lambda(e^{-\alpha u^2} - 1)}$$

1.7 - Example :

Student's law with n degrees of freedom is spherically invariant.

Student's law is the law of the ratio $\frac{X}{R}$ of two independent random variables X and R , the law of X being normal with mean 0 and variance σ^2 and that of R being a χ^2 with n degrees of freedom.

Thus the joint density f of X and R is given by :

$$f(x, r) = \frac{1}{\sqrt{2\pi}} \exp - \frac{x^2}{2\sigma^2} \frac{1}{\Gamma(\frac{n}{2}) 2^{\frac{n}{2}-1}} \exp - \frac{r^2}{2\sigma^2} \left(\frac{r}{\sigma}\right)^{n-1} \frac{1}{\sigma^2}$$

The characteristic function ϕ_T of $\frac{X}{R}$ can be computed directly. By definition one has

$$\begin{aligned} \forall u \in \mathbb{R} \quad \phi(u) &= E \left(e^{iu \frac{X}{R}} \right) \\ &= \int \int e^{iu \frac{x}{r}} \frac{1}{\sqrt{2\pi}} \exp - \frac{x^2}{2\sigma^2} \frac{1}{\Gamma(\frac{n}{2}) 2^{\frac{n}{2}-1}} \exp - \frac{r^2}{2\sigma^2} \left(\frac{r}{\sigma}\right)^{n-1} \frac{dx}{\sigma} \frac{dr}{\sigma} \end{aligned}$$

Applying Fubini's theorem, one gets, integrating over x,

$$\phi_T(u) = \int_{\mathbb{R}_+} \frac{1}{\Gamma(\frac{n}{2}) 2^{\frac{n}{2}-1}} e^{-\frac{u^2 \sigma^2}{2r^2}} \left(\frac{r}{\sigma}\right)^{n-1} e^{-\frac{r^2}{2\sigma^2}} \frac{dr}{\sigma}$$

To obtain the usual form, of the characteristic function set $a \triangleq \frac{\sigma}{r}$; this transformation is a homeomorphism of $]0, \infty[$, so that

$$\phi_T(u) = \int_{\mathbb{R}_+} e^{-\frac{u^2 a^2}{2}} \frac{1}{\Gamma(\frac{n}{2}) 2^{\frac{n}{2}-1}} e^{-\frac{1}{2a^2}} \frac{da}{a^{n+1}}$$

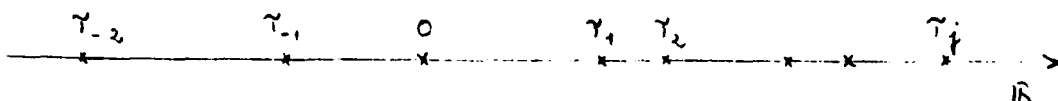
11 - THE CHARACTERISTIC FUNCTIONAL OF THE FILTERED POISSON PROCESS

Let C be a spherically invariant random variable. We have seen that its characteristic function has the representation :

$$(1) \quad \phi(u) \triangleq E(e^{iuC}) = \int_{\mathbb{R}_+} e^{-\frac{a^2 u^2}{2}} d\mu(a) \quad \text{where } \mu \text{ is a probability measure on } (\mathbb{R}_+, \mathcal{B}_+)$$

bility measure on $(\mathbb{R}_+, \mathcal{B}_+)$

Let τ_j be the time of the j -th jump of a stationary Poisson process with parameter λ .



As is well known for any fixed interval $[a, b]$ the number N of events τ_j in $[a, b]$ is a Poisson random variable with parameter $(b-a)$ so that

$$(2) \quad P(N = n) = e^{-\lambda(b-a)} \frac{[\lambda(b-a)]^n}{n!}$$

11.1 - Property :

When $N = n$ is fixed, the times $\tau_1, \tau_2, \dots, \tau_{N=n}$ of intervals $[a, b]$, are random and can be chosen independent and of same uniform law : $\frac{d\tau}{b-a}$.

Finally, a jump at time τ , of normalized sized, has an effect described by $G(t, \tau)$.

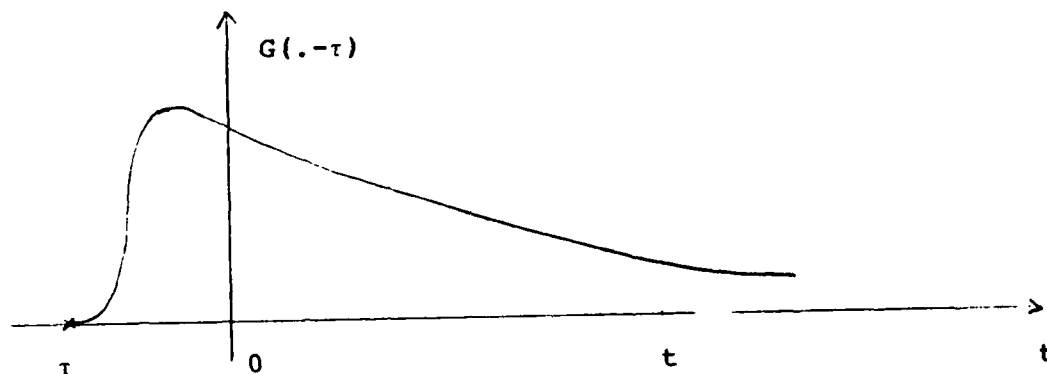
If we suppose the stationarity then

$$G(t, \tau) = G(t - \tau)$$

If we suppose the causality principle then :

$$(3) \quad G(t, \tau) = \begin{cases} 0 & \tau > t \\ G(t-\tau) & \tau \leq t \end{cases}$$

G is called the Green function of the process.



The filtered Poisson process is then defined by the relation :

$$(4) \quad \forall t \in \mathbb{R} \quad Z(t) \triangleq \sum_{j \in \mathbb{Z}} C_j(\omega) G(t - \tau_j)$$

The Poisson process and the amplitudes $C_j \quad j \in \mathbb{Z}$ are supposed to be independent.

In what follow, the following convenient notation shall be used :

$$(5) \quad \forall T \in \mathbb{R} \quad G_T(s) \triangleq G(s) 1_{[0, T]}(s)$$

$1_{[0, T]}$ being the indicator of the interval $[0, T]$

II.2 - Proposition : If

$$(6) \quad E(|C|) < \infty \text{ and}$$

$$(7) \quad \int_0^\infty |G(s)| ds < \infty, \text{ then}$$

$Z(t)$ is integrable and $E(|Z(t)|) \leq \lambda E(|C|) \int_0^\infty |G(s)| ds$

Proof : Since $|Z(t)| \leq \sum_j |C_j| |G(t - \tau_j)|$, one has

$$(8) \quad E(|Z(t)|) \leq E(|C|) E\left(\sum_{j \in \mathbb{Z}} |G(t-\tau_j)|\right)$$

An upper bound for $E\left(\sum_{j \in \mathbb{Z}} |G(t-\tau_j)|\right)$ is obtained by considering finite time effects represented by the expression :

$$(9) \quad E\left(\sum_{j \in \mathbb{Z}} |G_T(t - \tau_j)|\right)$$

$G_T(t - \tau_j)$ is zero except for τ_j in $[t - T, t]$, so that conditioning on the number of τ_j 's in $[t - T, t]$ one has using probabilities (2) and property (3) :

$$\begin{aligned} E\left(\sum_{j \in \mathbb{Z}} |G_T(t-\tau_j)|\right) &= \sum_{n=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \left(\sum_{j=1}^n E(|G_T(t-\tau_j)| / N = n) \right) \\ &= e^{-\lambda T} \sum_{n=1}^{\infty} \frac{(\lambda T)^n}{n!} n \int_{t-T}^t |G(t-\tau)| \frac{d\tau}{T} \\ &= \lambda \int_0^T |G(s)| ds, \text{ so that} \end{aligned}$$

$$\forall T \in \mathbb{R}_+ \quad E\left(\sum_{j \in \mathbb{Z}} |G_T(t-\tau_j)|\right) = \lambda \int_0^T |G(s)| ds$$

Taking the limit as $T \rightarrow \infty$, one has the needed result :

$$(10) \quad E\left(\sum_{j \in \mathbb{Z}} |G(t-\tau_j)|\right) = \lambda \int_0^{\infty} |G(s)| ds \quad \blacksquare$$

We are going to characterize the temporal law of process Z . To have a practical expression for the characteristic function of

$$(11) \quad \forall m \in \mathbb{N} \quad \forall t_1, t_2, \dots, t_m \in \mathbb{R} \quad Z_{\mathcal{C}} \triangleq (Z(t_1), Z(t_2), \dots, Z(t_m))$$

further assumptions are needed, as follows :

$$(12) \quad \int_0^{\infty} |G(s)|^2 ds < \infty \text{ and even } \int_0^{\infty} |G(s)|^2 ds = 1 \text{ and}$$

$$(13) \quad \int_0^{\infty} a^2 d\mu(a) < \infty$$

$$\text{Since } E(|C|^2) = \int_{\mathbb{R}_+} d\mu(a) \int_{\mathbb{R}} c^2 \frac{1}{\sqrt{2\pi} a} \exp - \frac{1}{2} \frac{c^2}{a^2} dc = \int_{\mathbb{R}_+} a^2 d\mu(a) < \infty$$

(13) means that C has finite variance.

$$11.3 - \text{Lemma} : \int_0^s |1 - \exp - \frac{a^2}{2} G^2(\sigma)| d\sigma \leq \frac{a^2}{2} \int_0^s G^2(\sigma) d\sigma$$

Proof : Consider $f(\lambda) \triangleq 1 - \exp - \lambda \frac{a^2}{2} G^2(\sigma)$ then

$$\frac{df}{d\lambda}(\lambda) = \frac{a^2}{2} G^2(\sigma) \exp - \lambda \frac{a^2}{2} G^2(\sigma) \text{ and}$$

$$\int_0^\lambda \frac{a^2}{2} G^2(\sigma) \exp - \lambda \frac{a^2}{2} G^2(\sigma) d\lambda = 1 - \exp - \lambda \frac{a^2}{2} G^2(\sigma)$$

At last, we take $\lambda = 1$ then

$$\begin{aligned} \left| 1 - \exp - \frac{a^2}{2} G^2(\sigma) \right| &\leq \int_0^1 \frac{a^2}{2} G^2(\sigma) \exp - \lambda \frac{a^2}{2} G^2(\sigma) d\lambda \\ &\leq \int_0^1 \frac{a^2}{2} G^2(\sigma) d\lambda = \frac{a^2}{2} G^2(\sigma) \quad \blacksquare \end{aligned}$$

11.4 - Theorem :

If Z is the derived Poisson process :

$$(4) \quad \forall t \in \mathbb{R} \quad Z(t) = \sum_{j \in \mathbb{Z}} C_j G(t - \tau_j)$$

if (14) and (15) are satisfied :

$$\int_{\mathbb{R}} |G(s)|^2 ds < \infty \text{ and } \int_{\mathbb{R}_+} a^2 d\mu(a) < \infty \text{ then}$$

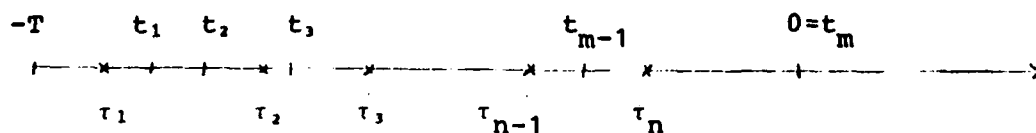
$$\forall m \in \mathbb{N} \quad \forall (t_1, t_2, \dots, t_m) \in \mathbb{R}^m \quad \forall u = (u_1, u_2, \dots, u_m) \in \mathbb{R}^m$$

$$(14) \quad \phi(u) \triangleq E(e^{i\langle u, Z \rangle}) =$$

$$\exp - \lambda \int_{\mathbb{R} \times \mathbb{R}_+} ds d\mu(a) \left[1 - \exp \left\{ - \frac{a^2}{2} \left(\sum_{k=1}^m u_k G(t_k + s) \right)^2 \right\} \right]$$

Proof : Without loss of generality, one may assume that

$$t_1 \leq t_2 \leq \dots \leq t_m = 0 \text{ and that } T > t_m - t_1$$



By definition of the inner product and of $z_{\mathcal{G}}$,

$$(15) \quad \langle u, z_{\mathcal{G}} \rangle \triangleq u_1 z(t_1) + \dots + u_m z(t_m) \\ = \sum_{j \in \mathbb{Z}} C_j \sum_{k=1}^m u_k G(t_k - \tau_j)$$

as done previously the calculation is performed first for finite time effect by conditioning on the number of jumps :

$$(16) \quad \langle u, z_{\mathcal{G}}^T \rangle = \sum_{j \in \mathbb{Z}} C_j \sum_{k=1}^m u_k G_T(t_k - \tau_j)$$

and we compute the characteristic function of this random variable.

So

$$(17) \quad \phi_T(u) \triangleq E(e^{i \langle u, z_{\mathcal{G}}^T \rangle}) \\ = e^{-\lambda T} + \sum_{n=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \frac{n}{\pi} E(e^{i \sum_{k=1}^n C_j u_k G_T(t_k - \tau_j)} / N = n) \\ = e^{-\lambda T} \sum_{n=1}^{\infty} \frac{(\lambda T)^n}{n!} \left[\int_{-T}^0 \frac{d\tau}{T} \int d\mu(a) e^{-\frac{a^2}{2} \left(\sum_k u_k G_T(t_k - \tau) \right)^2} \right]^n \\ = e^{-\lambda T} e^{+\lambda \int_{-T}^0 \int_{\mathbb{R}_+} d\tau d\mu(a) e^{-\frac{a^2}{2} \left(\sum_k u_k G_T(t_k - \tau) \right)^2}} \\ = \exp \left\{ -\lambda \int_0^T \int_{\mathbb{R}_+} d\tau d\mu(a) \left[1 - \exp \left\{ -\frac{a^2}{2} \left(\sum_k u_k G_T(t_k + \tau) \right) \right\} \right] \right\}$$

We will transform the exponent of the exponential ; let :

$$(18) \quad \psi_T(u) \triangleq \log \phi_T(u) = -\lambda \int_0^T \int_{\mathbb{R}_+} ds d\mu(a) \left[1 - e^{-\frac{a^2}{2} \left(\sum_k u_k G_T(t_k + s) \right)^2} \right]$$

and :

$$(19) \quad G_G(s) \triangleq \sum_{k=1}^m u_k G_T(t_k - s)$$

Then by lemma II.3 :

$$(20) \quad \int_{\mathbb{R}_+^2} ds d\mu(a) \left[1 - \exp \left\{ - \frac{a^2}{2} G^2(s) \right\} \right] \leq \int_{\mathbb{R}_+^2} ds d\mu(a) \frac{a^2}{2} G_G^2(s)$$

$$\leq \int_{\mathbb{R}_+^2} ds d\mu(a) \frac{a^2}{2} \sum_{k=1}^m u_k^2 \sum_{k=1}^m G_T(t_k + s)^2$$

$$\leq \int_{\mathbb{R}_+^2} ds d\mu(a) \frac{a^2}{2} \sum_{k=1}^m G(t_k + s)^2 \sum_{k=1}^m u_k^2 < \infty$$

So by the dominated convergence

$$\begin{aligned} \lim_{T \rightarrow \infty} \int_0^T \int_0^\infty ds d\mu(a) \left[1 - \exp \left\{ - \frac{a^2}{2} \left(\sum_{k=1}^m u_k G_T(t_k + s) \right)^2 \right\} \right] \\ = \int_0^\infty \int_0^\infty ds d\mu(a) \left[1 - \exp \left\{ - \frac{a^2}{2} \left(\sum_{k=1}^m u_k G(t_k + s) \right)^2 \right\} \right] \end{aligned}$$

So we obtain :

$$\begin{aligned} \phi(u) &= \exp - \lambda \int_{\mathbb{R}_+^2} ds d\mu(a) \left[1 - \exp - \frac{a^2}{2} \left(\sum_{k=1}^m u_k G(t_k + s) \right)^2 \right] \\ &= \exp - \lambda \int_{\mathbb{R}_+} d\mu(a) \int_{-\infty}^{+\infty} ds \left[1 - \exp - \frac{a^2}{2} \left(\sum_{k=1}^m u_k G(t_k + s) \right)^2 \right] \end{aligned}$$

because, for $t_k + s < 0$ or $s < -t_k$ and so for $s < 0$, one has $G(t_k + s) = 0$.

Furthermore since any translation on s does not change the integral the assumption $t_v \triangleq t_1 \vee t_2 \vee \dots \vee t_m = 0$ is not a restriction. ■

Remark : The process Z as defined is real valued. It is however very easy to define a vector valued process as follows :

$$(21) \quad \underline{Z}(t) = \sum_{j=-\infty}^{+\infty} C_j \underline{G}(t-\tau_j) \in \mathbb{R}^q$$

Where \underline{G} is a function with values in \mathbb{R}^q .

To obtain the characteristic function of Z , it is useful to introduce some new notation.

Let u be the $(q \times n)$ matrix with entries $u_{i,j}$ and Z be the matrix with entries $Z_i(t_j)$.

Then, $\langle u, Z \rangle \triangleq \text{trace } u^t Z$ is the Hilbert-Schmidt inner product for matrices.

One can then state :

11.5 - Theorem :

If \underline{Z} is a filtered Poisson process with values in \mathbb{R}^q

$$(22) \quad \forall t \in \mathbb{R} \quad \underline{Z}(t) = \sum_{j \in \mathbb{Z}} C_j \underline{G}(t - \tau_j)$$

and if $\int_{\mathbb{R}} \|\underline{G}(s)\|^2 ds < \infty$ and $\int_{\mathbb{R}_+} a^2 d\mu(a) < \infty$ then

$$\forall m \in \mathbb{N} \quad \forall (t_1, t_2, \dots, t_m) \in \mathbb{R}^m \quad \forall u \in \mathbb{R}^{q \times m}$$

$$(23) \quad \phi(u) \triangleq E(e^{i\langle u, Z \rangle}) =$$

$$\exp - \lambda \int ds d\mu(a) \left[1 - \exp - \frac{a^2}{2} \left(\sum_{k=1}^m \langle u_k, \underline{G}(t_k + s) \rangle \right)^2 \right]$$

Remarks : Formulae 23 and 14 yield the same result if in (14),

$u_k \underline{G}(t_k + s)$ is replaced by

$$\langle u_k, \underline{G}(t_k + s) \rangle = \sum_{i=1,1,\dots,q} u_{i,k} G_i(t_k + s)$$

with the convention that $\|\underline{G}(s)\|^2 \triangleq \langle \underline{G}, \underline{G} \rangle = \text{Trace } \underline{G}^t \underline{G}$.

III - CALCULATION OF THE MOMENTS OF Z

A new process Z with characteristic function

$$\phi(u) = E(e^{i(u_1 Z(t_1) + \dots + u_m Z(t_m))}) = E(e^{i\langle u, Z \rangle})$$

$$(1) \quad = \exp - \lambda \int_{\mathbb{R} \times \mathbb{R}_+} ds d\mu(a) \left[1 - \exp - \frac{a^2}{2} \left(\sum_k u_k G(t_k + s) \right)^2 \right]$$

is useful if it is different from the usual ones, Poisson processes, spherically invariant processes, and more accurate to some applications.

The aim of this section is to obtain the moments of Z. To that end it is assumed that Z has moments of all orders and that ϕ is analytical at the origin.

$$(2) \quad \begin{aligned} \phi(u) &= E \left(\sum_{k=0}^{\infty} \frac{i^k}{k!} (u_1 Z(t_1) + \dots + u_m Z(t_m))^k \right) \\ &= \sum_{k=0}^{\infty} \frac{i^k}{k!} E(\langle u, Z \rangle^k) \end{aligned}$$

The explicit expression for ϕ shows that $\phi(u)$ is a function of the square of $\langle u, G \rangle(s)$ defined by :

$$(3) \quad \langle u, G \rangle(s) \triangleq \sum_k u_k G(t_k + s)$$

III.1 - Lemma : The odd moments of Z are equals to zero.

Proof : The series expansion of the exponential function yields :

$$(4) \quad 1 - \exp - \frac{a^2}{2} \langle u, G \rangle^2(s) = - \sum_{p=1}^{\infty} \left(- \frac{a^2}{2} \right)^p \frac{\langle u, G \rangle^{2p}(s)}{p!}$$

so that

$$(5) \quad \phi(u) = \exp + \lambda \int_{\mathbb{R} \times \mathbb{R}_+} ds d\mu(a) \sum_{p=1}^{\infty} \left(- \frac{a^2}{2} \right)^p \frac{\langle u, G \rangle^{2p}(s)}{p!}$$

Finally this last exponential can be expanded to yield a new series in the variable $\int_{\mathbb{R}_+} ds \langle u, G_{\epsilon} \rangle^2 (s)$ where each term is homogeneous in u and of even degree. The odd moments which are the coefficients accompanying expression of the form $u_1^{i_1} u_2^{i_2} \dots u_m^{i_m}$ with $i_1 + i_2 + \dots + i_m$ odd vanish. ■

It is now convenient to introduce the second characteristic function $\psi(u)$:

$$(6) \quad \psi(u) \triangleq \log \phi(u) = - \lambda \int_{\mathbb{R}_+^2} ds d\mu(a) \left[1 - \exp - \frac{a^2}{2} \langle u, G_{\epsilon} \rangle^2 (s) \right]$$

$$III.2 - \text{Lemma} : \quad \psi(u) = - \sum_{q=1}^{\infty} \frac{(1-\phi(u))^q}{q}$$

Proof : $\psi(u) = \log \phi(u) = \log [1 - (1-\phi(u))]$ and for $0 < \epsilon < 1$

$$\log (1 - \epsilon) = - \left(\epsilon + \frac{\epsilon^2}{2} + \dots + \frac{\epsilon^q}{q} + \dots \right) \quad \blacksquare$$

$$III.3 - \text{Lemma} : \quad \psi(u) = - \sum_{q=1}^{\infty} \frac{1}{q} \left[\sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{(2\ell)!} E (\langle u, Z \rangle^{2\ell}) \right]^q$$

Proof : Since the odd moments of Z_{ϵ} vanish, the expansion of ϕ yields :

$$(7) \quad \phi(u) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell)!} E (\langle u, Z_{\epsilon} \rangle^{2\ell}) \quad \text{so that}$$

$$(8) \quad 1 - \phi(u) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{(2\ell)!} E (\langle u, Z_{\epsilon} \rangle^{2\ell}) \quad \blacksquare$$

Comparing the two expressions for ψ given respectively by Lemma III.3 and the relation (5), one obtains the moments of Z . One has :

$$(9) \quad - \sum_{q=1}^{\infty} \frac{1}{q} \left[\sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{(2\ell)!} E(\langle u, Z_{\zeta} \rangle^{2\ell}) \right]^q = \psi(u) =$$

$$\lambda \sum_{p=1}^{\infty} \frac{1}{p!} \int_{\mathbb{R}_+} d\mu(a) \left(-\frac{a^2}{2}\right)^p \int_{\mathbb{R}} ds \langle u, G_{\zeta} \rangle^{2p}(s)$$

We consider first the coefficients of $u_1^{i_1} u_2^{i_2}$ with $i_1 + i_2 = 2$ to compute moments of second order.

III.4 - Proposition : $\forall t \quad E(Z(t)^2) = \lambda \int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}} G(s)^2 ds$

$$\forall t_1, t_2 \in \mathbb{R} \quad E(Z(t_1)Z(t_2)) = \lambda \int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}} G(t_1+s) G(t_2+s) ds$$

Proof : One must have

$$- \frac{1}{1} \frac{(-1)^{1+1}}{2!} E(\langle u, Z_{\zeta} \rangle^2) =$$

$$\lambda \frac{1}{1!} \int d\mu(a) \left(-\frac{a^2}{2}\right) \int ds \langle u, G_{\zeta} \rangle^2(s)$$

Using the respective definitions of $\langle u, Z_{\zeta} \rangle$ (II.15) and $\langle u, G_{\zeta} \rangle$ (3) one obtains

$$(10) \quad E(u_1 Z(t_1) + u_2 Z(t_2))^2 =$$

$$\lambda \int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}} ds (u_1 G(t_1+s) + u_2 G(t_2+s))^2 \text{ and equating}$$

the coefficients of u_1^2 , u_2^2 and $u_1 u_2$:

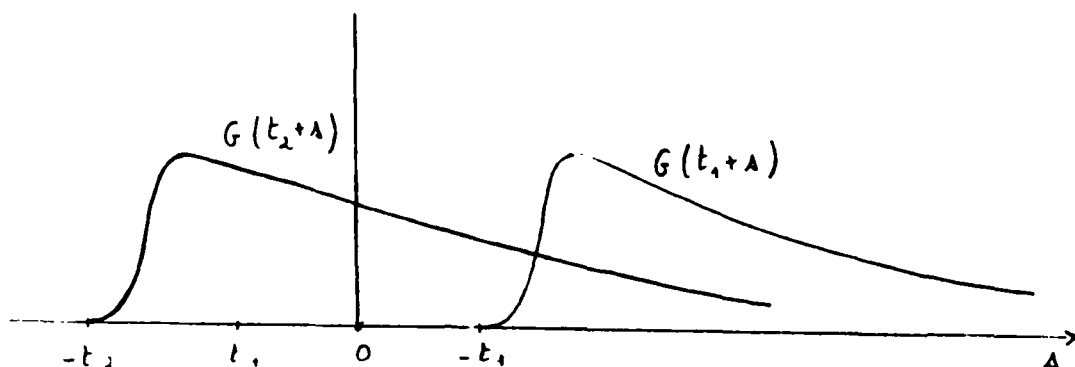
$$E(Z(t_1)^2) = \lambda \int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}} ds G(t_1+s)^2 =$$

$$\lambda \int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}} ds G(s)^2 = E(Z(t_2)^2)$$

and

$$E(Z(t_1) Z(t_2)) = \lambda \int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}} ds G(t_1+s) G(t_2+s) \quad \blacksquare$$

Remark :



If $t_1 < t_2$, the integrals $\int_{\mathbb{R}} ds G(t_1+s) G(t_2+s)$ and $\int_0^{\infty} ds G(s) G(t_2-t_1+s)$ are equal.

The next proposition lists the moments of fourth order.

III.5 - Proposition

$$1) E(Z(t^4)) = 3\lambda \int_{\mathbb{R}_+} a^4 d\mu(a) \int_{\mathbb{R}_+} ds G^4(s) + 3\lambda^2 \left[\int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}_+} ds G^2(s) \right]^2$$

$$2) E(Z(t_1)^2 Z(t_2)^2) = 3\lambda \int_{\mathbb{R}_+} a^4 d\mu(a) \int_{\mathbb{R}} ds G^2(t_1+s) G^2(t_2+s)$$

$$+ \lambda^2 \left[\int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}_+} ds G(s)^2 \right]^2$$

$$+ 2 \lambda^2 \left[\int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}} ds G(t_1+s) G(t_2+s) \right]^2$$

$$3) E(Z(t_1) Z(t_2) Z(t_3) Z(t_4))$$

$$= 3\lambda \int_{\mathbb{R}_+} a^4 d\mu(a) \int_{\mathbb{R}} ds G(t_1+s) G(t_2+s) G(t_3+s) G(t_4+s)$$

$$+ \frac{1}{2} \lambda^2 \left[\int_{\mathbb{R}_+} a^2 d\mu(a) \right]^2 \left[\int_{\mathbb{R}} ds G(t_1+s) G(t_j+s) \int_{\mathbb{R}} ds G(t_k+s) G(t_l+s) \right]$$

where the summation is over all permutations (i, j, k, l) of (1, 2, 3, 4) such that $i < j$ and $k < l$.

Proof : The first terms in the expansions of ψ , formula (9) are respectively, for :

(11)

$$q = 1 \quad - \frac{1}{1} \left[\frac{(-1)^{1+1}}{2!} E(\langle u, Z_{\gamma} \rangle^2) + \frac{(-1)^{2+1}}{4!} E(\langle u, Z_{\gamma} \rangle^4) + \dots \right]^1 +$$

$$q = 2 \quad - \frac{1}{2} \left[\frac{(-1)^{1+1}}{2!} E(\langle u, Z_{\gamma} \rangle^2) + \frac{(-1)^{2+1}}{4!} E(\langle u, Z_{\gamma} \rangle^4) + \dots \right]^2 + \dots$$

Equating the homogeneous terms of degree 4 in u , one obtains :

$$(12) \quad \frac{1}{4!} E(u_1 Z(t_1) + u_2 Z(t_2) + u_3 Z(t_3) + u_4 Z(t_4))^4 +$$

$$(13) \quad - \frac{1}{2} \left[\frac{1}{2!} E(u_1 Z(t_1) + u_2 Z(t_2) + u_3 Z(t_3) + u_4 Z(t_4))^2 \right]^2 =$$

$$(14) \quad \lambda \frac{1}{2!} \int \frac{a^4}{4} d\mu(a) \int ds \left[u_1 G(t_1+s) + u_2 G(t_2+s) + u_3 G(t_3+s) + u_4 G(t_4+s) \right]^4$$

This yields, if $t_i = t$, $i = 1, 2, 3, 4$

$$\frac{1}{4!} E(Z(t)^4) = \frac{1}{2 \times 4} \left[E(Z(t)^2) \right]^2$$

$$+ \lambda \frac{1}{2 \times 4} \int a^4 d\mu(a) \int ds G^4(s)$$

The first term on the right hand side is given by

Proposition III.4

In the case of $t_1 = t, < t_2 = t,$

$$\begin{aligned} & \frac{1}{4!} C_4^2 E(Z(t_1)^2 Z(t_2)^2) = \\ & \frac{1}{2} \left(\frac{1}{2}\right)^2 \left[2E(Z(t_1)^2)E(Z(t_2)^2) + 2^2 (E(Z(t_1)Z(t_2)))^2 \right] + \\ & \lambda \frac{1}{2!} C_4^2 \int \frac{a^4}{4} d\mu(a) \int ds G^2(t_1+s) G^2(t_2+s) \end{aligned}$$

where $C_4^2 = \frac{4!}{(2!)^2}$. Again the first term on the right hand side is given by Proposition III.4.

The case of $t_1 < t_2 < t_3 < t_4$ is more complicated. For any random variables X_i and real number u_i , $i = 1, 2, 3, 4$, one has that :

$$\begin{aligned} (15) \quad & \left[E(u_1 X_1 + u_2 X_2 + u_3 X_3 + u_4 X_4)^2 \right]^2 = \\ & \left[E(u_1^2 X_1^2 + u_2^2 X_2^2 + u_3^2 X_3^2 + u_4^2 X_4^2) + \right. \\ & \quad 2E(u_1 u_2 X_1 X_2 + u_1 u_3 X_1 X_3 + u_1 u_4 X_1 X_4 \\ & \quad \quad + u_2 u_3 X_2 X_3 + u_2 u_4 X_2 X_4 \\ & \quad \quad \quad \left. + u_3 u_4 X_3 X_4) \right]^2 \\ & = \sum_i u_i^4 [E(X_i^2)]^2 \\ & \quad + 4 \sum_{i < j} u_i^2 u_j^2 [E(X_i X_j)]^2 \\ & \quad + 2 \sum_{i < j} u_i^2 u_j^2 [E(X_i^2) E(X_j^2)] \\ & \quad + 4 \sum_{\substack{i < j \\ \text{and } (i,j) \neq (k,l)}} u_i u_j u_k u_l [E(X_i X_j) E(X_k X_l)] \\ & \quad + 2 \sum_i u_i^2 u_k u_l [E(X_i^2) E(X_k X_l)] \end{aligned}$$

The pair $[(i < j), (k < l)]$ is not ordered so that there are 6×5 such pairs. The same is true for the pairs $[(i), (k < l)]$.

One then equates the terms containing $u_i u_j u_k u_l$ with (i, j, k, l) permutation of $(1, 2, 3, 4)$ and $i < j < k < l$; there are only six such permutations and

$$\frac{1}{4!} 4! E(Z(t_1) Z(t_2) Z(t_3) Z(t_4)) =$$

$$\frac{4}{2 \times 2^2} \left[\sum_{\substack{i < j, k < l \\ (i, j, k, l) \text{ a permutation}}} E(Z(t_i) Z(t_j)) E(Z(t_k) Z(t_l)) \right] +$$

$$\lambda \frac{1}{2} \int_{\mathbb{R}^+} \frac{a^4}{4} d\mu(a) \int_{\mathbb{R}} ds 4! G(t_1+s) G(t_2+s) G(t_3+s) G(t_4+s)$$

For the summation $\sum_{\substack{i < j, k < l \\ \text{and } (i, j) \neq (k, l)}}$ one choose $i < j$ among $1, 2, 3, 4$

and $k < l$ are the two remaining integers. ■

Remark : One could also compute other moments such as

$E(Z(t_1) Z(t_2)^2 Z(t_3))$ and $E(Z(t_1) Z(t_2)^3)$ but these are of a lesser interest.

IV - IDENTIFICATION OF THE GREEN FUNCTION G

Given some experimental data, one may assess the adequacy of the model and in case it is satisfactory, to estimate the parameters which are the Green function G, the parameter λ of the original Poisson process and the mixing law μ for the amplitude C. In this section, an estimation procedure for G is presented. Estimation of λ and μ is the subject of the next one.

Statistical estimation based on one trajectory of the observed process requires that the process be strongly stationary which is the case for the process Z. Its characteristic function (theorem II.4) is indeed invariant under translation of the time variables.

The covariance Γ of $X(t)$ and Γ_2 of $X^2(t)$ are stationary and given by the relation :

$$(1) \quad \Gamma(\tau) \triangleq E(Z(t)Z(t-\tau)) = \lambda \int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}} G(s) G(s+\tau) ds$$

$$(2) \quad \Gamma_2(\tau) \triangleq E(Z^2(t) Z^2(t-\tau)) =$$

$$3\lambda \int_{\mathbb{R}_+} a^4 d\mu(a) \int_{\mathbb{R}} ds G^2(s) G^2(s+\tau) + \left[\lambda \int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}} ds G^2(s) \right]^2$$

$$+ 2\lambda^2 \left[\int_{\mathbb{R}_+} a^2 d\mu(a) \int_{\mathbb{R}} ds G(s) G(s+\tau) \right]^2$$

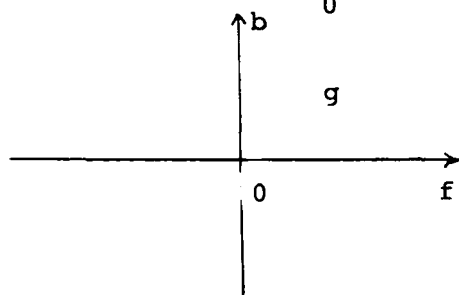
In (1), $\lambda \int_{\mathbb{R}_+} a^2 d\mu(a)$ is a normalization factor and, assuming (II.12) one has $\Gamma(0) = \lambda \int_{\mathbb{R}_+} a^2 d\mu(a)$.

IV.1 - Proposition : Let $g(f) \triangleq \int_{-\infty}^{+\infty} e^{+isf} G(s) ds = \mathcal{T}(G)$. Then

$$(3) \quad \Gamma(\tau) = \Gamma(0) \int_{-\infty}^{+\infty} e^{-isf} |g(f)|^2 \frac{df}{2\pi}$$

g is the Fourier transform \mathcal{F} of G , hence the notation $g = \mathcal{F}(G)$. It is square integrable and vanishes for $s < 0$; one defines (cf [3], p. 30) its analytical extension $g(f + ib)$ by the relation

$$g(f + ib) \triangleq \int_0^{\infty} e^{is(f+ib)} G(s) ds \quad b > 0$$



g is said to belong to the Hardy space H^{2+} .

Proof : Taking the inverse Fourier transform \mathcal{F}^{-1} of g , one gets :

$$(4) \quad G(s) = \mathcal{F}^{-1}(g)(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isf} g(f) df$$

In (1) the integral in s , is in fact, a convolution in s , let

$$(5) \quad G'(r) \triangleq G(-r) \quad \text{then}$$

$$(6) \quad \int_{\mathbb{R}} G(s) G(s+\tau) ds = \int_{\mathbb{R}} G(s-\tau) G(\tau) d\tau = \int_{\mathbb{R}} G'(\tau-s) G(s) ds = G' \star_s G(\tau)$$

With self-evident notation, we know that :

$$G' \star_s G = \mathcal{F}^{-1} (\mathcal{F} G' \cdot \mathcal{F} G)$$

But, by definition $g(f) \triangleq \mathcal{F} G(f) = \int_{-\infty}^{+\infty} e^{isf} G(s) ds$,

and $\overline{g}(f) = \mathcal{F} G'(f) = \int_{-\infty}^{+\infty} e^{isf} G(-s) ds$, consequently

$$(7) \quad \Gamma(\tau) = \Gamma(0) \left[G' \star_s G \right](\tau) =$$

$$\Gamma(0) \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isf} |g(f)|^2 df$$

This result shows in particular that the gain $|g(f)|$ of the causal filter G can be estimated when an estimate of the

covariance Γ is available.

It is a standard procedure in automatic control to recover the response G of a causal filter from its gain $|g|$.

There is however a difficulty which can be illustrated with the example of a rational filter G whose Fourier transform is given by the relation :

$$(8) \quad g(f) = \frac{(f - b_1)(f - b_2) \dots (f - b_q)}{(f - a_1)(f - a_2) \dots (f - a_p)} \quad p, q \in \mathbb{N}$$

Since G is causal, g is entire in the upper half-plane \mathbb{R}_+ so that the poles a_1, a_2, \dots, a_p of g belong to the lower half plane \mathbb{R}_- .

Since $G(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ifs} g(f) df$ is assumed real and then

$$G(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{+ifs} \overline{g(f)} df = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ifs} \overline{g(-f)} df \quad \text{so}$$

$$(9) \quad g(f) = \overline{g(-f)}$$

(9) shows that b and $-b$ are simultaneously zeroes of g and a and $-a$ are simultaneously poles of g .

Let b_i be a zero of g and define g' by the relation :

$$g'(f) \triangleq \frac{(f - b_1) \dots (f - b_q)}{(f - a_1) \dots (f - a_p)} \times \frac{f - \bar{b}_i}{f - b_i}$$

g' is a frequency response such that $|g'(f)| = |g(f)|$. If f is real, $\frac{f - \bar{b}_i}{f - b_i}$ has modulus 1 and is called an interior function (cf [3], p. 36).

Multiplying (8) by $\frac{f - \bar{b}_i}{f - b_i} \times \frac{f + \bar{b}_i}{f + b_i}$, one obtains the frequency response of a causal real filter.

Thus $|g|$ does not uniquely determine G and the question arises as to the best choice of G .

IV.2 - Definition : A causal filter G is said to have a minimal phase if $\frac{1}{g}$ is the frequency response of a causal filter.

For a filter with minimal phase, the zeroes b of g have to be in the lower half plane and the following result obtains ([3] p. 37).

IV.3 - Proposition : The gain $|g|$ determines the causal, phase minimal filter G .

In fact, the frequency response g of this phase minimal filter G has an analytical extension in the upper half plane given by

$$(10) \quad g(f+ib) = \exp \frac{1}{i\pi} \int \frac{\gamma(f+ib) + 1}{\gamma - (f+ib)} \frac{\log |g(\gamma)|}{\gamma^2 + 1} d\gamma$$

This expression may be used to approximate $a(f)$,

$f \in \mathbb{R}$; for

$$\begin{aligned} \log g(f+ib) &= \frac{b}{\pi} \int \frac{\log |g(\gamma)| d\gamma}{(\gamma-f)^2 + b^2} \\ &+ \frac{i}{\pi} \int \left[\frac{(f-\gamma)}{(\gamma-f)^2 + b^2} + \frac{\gamma}{\gamma^2 + 1} \right] \log |g(\gamma)| d\gamma \end{aligned}$$

and thus

$$i) \quad \frac{b}{\pi} \int \frac{\log |g(\gamma)|}{(\gamma-f)^2 + b^2} d\gamma \xrightarrow{\text{a.s.}} \log |g(f)| \quad ([3] \text{ p. 37})$$

$$ii) \quad \frac{i}{\pi} \int \left[\frac{(f-\gamma)}{(\gamma-f)^2 + b^2} + \frac{\gamma}{\gamma^2 + 1} \right] \log |g(\gamma)| d\gamma \xrightarrow{\text{a.s.}} \text{Arg } g(f) \quad ([3] \text{ p. 38 and 51})$$

When b small this last integral gives an approximation of the phase $\text{Arg } g(f)$ of the uniquely determined, causal, minimal phase filter G .

G can be also approximated directly. For $b > 0$, define G^b by the relation

$$(11) \quad G^b(t) \triangleq \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ift} g(f+ib) df$$

Since $g(\cdot + ib)$ converges in $L^2(\mathbb{R}, \mathbb{R}, df)$ to $g(\cdot)$ as b goes to zero ([3] p. 30) and since the Fourier transform is an isometry in $L^2(\mathbb{R}, \mathbb{R}, df)$

$$(12) \quad G^b(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} df e^{-ift} \exp \left[\frac{1}{i\pi} \int \frac{\gamma(f+ib)+1}{\gamma-(f+ib)} \frac{\log |g(\gamma)|}{\gamma^2+1} \right] d\gamma$$

converges in $L^2(\mathbb{R}, \mathbb{R}, df)$ to $G(t)$, as b goes to zero.

$$(13) \quad \int \frac{\log |g(\gamma)|}{\gamma^2 + 1} d\gamma \text{ exists for functions } g \text{ in } H^{2+}.$$

The desired approximation G^b of G requires that the Fourier transform of a complex function be obtained.

In conclusion, the correlation function $\Gamma(\tau)$ determines the causal, phase minimal filter G which can be estimated.

The same is true for $\Gamma_2(\tau)$ as defined in (2) :

$$\begin{aligned} \Gamma_2(\tau) = 3\lambda \int_{\mathbb{R}_+} a^4 d\mu(a) \int_{\mathbb{R}} ds G^2(s) G^2(s+\tau) \\ + \Gamma(0)^2 + 2\Gamma(\tau)^2 \end{aligned}$$

Up to a normalization factor $3\lambda \int_{\mathbb{R}_+} a^4 d\mu(a)$, one has an estimate of $\int_{\mathbb{R}} ds G^2(s) G^2(s+\tau)$. This function of τ , also yields the gain, denoted $|g_2(\cdot)|$ of the causal filter G^2 and one may compute from this gain $|g_2(\cdot)|$, the response G^2 of a causal, phase minimal up to a normalization factor.

If the two estimations of G^2 , the first based on the moments $\Gamma(\tau) = E(Z(t) Z(t-\tau))$ and the second based on the moments $\Gamma_2(\tau) = E(Z^2(t) Z^2(t-\tau))$, are equal, nearly equal, one can safely accept that G is indeed a deterministic function.

V - IDENTIFICATION OF λ AND μ

Here the Green function G is supposed known and the aim is to estimate the unknown parameter λ of the underlying Poisson process as well as the mixing law μ determining the law of C .

Having assumed in part III that the characteristic function ϕ of Z can be expressed as a known series whose coefficients are the moments of Z , the relation

$$\psi(u) = \text{Log } \phi(u) = \sum_{p=1}^{\infty} u^{2p} \frac{(-1)^p}{2^p p!} \lambda \int_{\mathbb{R}_+} a^{2p} d\mu(a) \int_{\mathbb{R}} G(s)^{2p} ds$$

should yield, after proper truncation, estimate of λ and μ in terms of the moments of Z .

The problem can thus be stated as that of determining λ and μ given the equalities

$$(1) \quad \lambda \int_{\mathbb{R}_+} a^{2p} d\mu(a) = \alpha_p \quad p = 1, 2, \dots, l$$

where α_p is known, α_p is obtained by dividing the coefficient of u^{2p} in the expansion of ψ by $(-1)^p 2^p p! \int_{\mathbb{R}} G(s)^{2p} ds$

Such a problem can be restated as a classical moment problem ([1]). Indeed if $\tilde{\mu}$ is symmetric probability measure on $(\mathbb{R}, \mathcal{B})$ such that $\tilde{\mu}/\mathbb{R}_+ = \frac{1}{2} \mu$ the problem at hand becomes that of finding $\tilde{\mu}$ such that $\forall k \in \mathbb{N}$:

$$\int_{\mathbb{R}} a^{2k} d\tilde{\mu} = \int_{\mathbb{R}_+} a^{2k} d\mu = \alpha_k$$

$$\int_{\mathbb{R}} a^{2k+1} d\tilde{\mu} = 0$$

The procedure is as follows. Orthonormalise the sequence of functions $\{a^k ; k \in \mathbb{N} \cup \{0\}\}$ in the Hilbert space $L^2(\mathbb{R}, \mathcal{B}, \tilde{\mu})$ in order to obtain a sequence of orthonormal polynomials $\{P_k ; k \in \mathbb{N} \cup \{0\}\}$.

We recall the results about these orthogonal Polynomials as shown in [1], chap. I :

Let :

$$(2) \quad s_k \triangleq \int a^k d\tilde{\mu}(a) \text{ moment of order } k \text{ of } \tilde{\mu}$$

$$(3) \quad D_k \triangleq \begin{vmatrix} s_0 & s_1 & \dots & s_k \\ s_1 & s_2 & \dots & s_{k+1} \\ \dots & \dots & \dots & \dots \\ s_k & s_{k+1} & \dots & s_{2k} \end{vmatrix} \quad || \text{ denoted the determinant}$$

For the family $\{P_k ; k \in \mathbb{N} \cup \{0\}\}$, these relations are valid :

$$(4) \quad P_k(a) a^h = \frac{1}{\sqrt{D_{k-1} D_k}} \begin{vmatrix} s_0 & s_1 & \dots & s_k \\ s_1 & s_2 & \dots & s_{k+1} \\ \dots & \dots & \dots & \dots \\ a^h & a^{h+1} & \dots & a^{h+k} \end{vmatrix} \quad \text{and}$$

$$(5) \quad aP_k(a) = b_{k-1} P_{k-1}(a) + a_k P_k(a) + b_k P_{k+1}(a) \quad \text{where}$$

$$(6) \quad b_{-1} = 0$$

$$(7) \quad a_k = \int_{\mathbb{R}} a |P_k(a)|^2 d\tilde{\mu}(a)$$

$$(8) \quad b_k = \frac{\sqrt{D_{k-1} D_{k+1}}}{D_k}$$

V.1 - Proposition : If $\tilde{\mu}$ is a symmetric probability measure on $(\mathbb{R}, \mathcal{B})$ then

$$\forall k \in \mathbb{N} \quad \forall a \in \mathbb{R}$$

$$P_{2k}(-a) = P_{2k}(a)$$

$$P_{2k+1}(-a) = -P_{2k+1}(a)$$

Proof : $P_0(a) = 1$ and one proceed by induction.

Since a and $P_{2k}(a)$ are orthogonal by construction

$$a_{2k} = \int a |P_{2k}(a)|^2 d\mu(a) = 0, \text{ so that, by (5),}$$

$$a P_{2k}(a) - b_{2k-1} P_{2k-1}(a) = b_{2k} P_{2k+1}(a) \text{ and}$$

$$P_{2k+1}(-a) = \frac{1}{b_{2k}} \left[-a P_{2k}(-a) - b_{2k-1} P_{2k-1}(-a) \right] = -P_{2k+1}(a)$$

$$\text{Similarly } a_{2k+1} = \int a [P_{2k+1}(a)]^2 d\tilde{\mu}(a) = 0 \text{ so that, by (5),}$$

$$a P_{2k+1}(a) = b_{2k} P_{2k}(a) + b_{2k+1} P_{2k+2}(a) \text{ and hence}$$

$$P_{2k+2}(-a) = P_{2k+2}(a) \quad \blacksquare$$

V.2 - Corollary

$$P_{2k}(a) = \frac{1}{\sqrt{D_{2k} D_{2k-1}}} \begin{vmatrix} 1 & s_2 & \dots & s_{2k} \\ s_2 & s_4 & \dots & s_{2k+2} \\ \dots & \dots & \dots & \dots \\ 1 & a^2 & & a^{2k} \end{vmatrix}$$

$$P_{2k+1}(a) = \frac{1}{\sqrt{D_{2k+1} D_{2k}}} \begin{vmatrix} 1 & s_2 & \dots & s_{2k} \\ s_2 & s_4 & \dots & s_{2k+2} \\ \dots & \dots & \dots & \dots \\ a & a^3 & \dots & a^{2k+1} \end{vmatrix} = \sqrt{\frac{D_{2k-1}}{D_{2k+1}}} a P_{2k}(a)$$

Proof : One integrates $a^l P_{2k}(a)$ with respect to $\tilde{\mu}$ and clearly

$P_{2k}(a)$ is orthogonal to $1, a, \dots, a^{2k-1}$ and similarly

$P_{2k+1}(a)$ is orthogonal to $1, a, \dots, a^{2k}$. \blacksquare

Since only a finite number of moments is available, μ can be determined if it is a finite convex combination of point masses.

Let thus μ be defined by the relation :

$$(9) \quad \mu = \mu_1 \delta_{\sigma_1} + \mu_2 \delta_{\sigma_2} + \dots + \mu_\ell \delta_{\sigma_\ell} \quad \text{where}$$

$$(10) \quad \mu_1 + \mu_2 + \dots + \mu_\ell = 1 \quad \mu_1 > 0 \dots \mu_\ell > 0.$$

The equalities $\lambda \int_{\mathbb{R}_+} a^{2p} d\mu = \alpha_p$ then become :

$$(11) \quad \begin{aligned} \lambda (\mu_1 \sigma_1^2 + \mu_2 \sigma_2^2 + \dots + \mu_\ell \sigma_\ell^2) &= \alpha_1 \\ \lambda (\mu_1 \sigma_1^{2 \cdot 2} + \mu_2 \sigma_2^{2 \cdot 2} + \dots + \mu_\ell \sigma_\ell^{2 \cdot 2}) &= \alpha_2 \\ &\vdots \\ \lambda (\mu_1 \sigma_1^{2 \cdot 2\ell} + \mu_2 \sigma_2^{2 \cdot 2\ell} + \dots + \mu_\ell \sigma_\ell^{2 \cdot 2\ell}) &= \alpha_{2\ell} \end{aligned}$$

There are as many equations as there are unknowns.

V.3 - Remarks : When $\tilde{\mu}$ is symmetric, the odd moments vanish. One writes 4ℓ equalities but 2ℓ comes from odd moments or from symmetry.

The last equality for the odd moment is $\int a^{4\ell-1} d\tilde{\mu}(a) = 0$.

The last relation $\lambda \int_{\mathbb{R}_+} a^{4\ell} d\tilde{\mu}(a) = \alpha_{2\ell}$ does not appear in the classical moment problem.

We propose to compute μ from the 2ℓ first relations and after to obtain λ from the last one :

$$\lambda \int_{\mathbb{R}_+} a^{4\ell} d\mu(a) = \alpha_{2\ell}$$

In fact, we will eliminate the unknowns μ_i, σ_i $i = 1, 2, \dots, \ell$ of the system (10) (11) to obtain λ directly from the α !

Once λ is known, one has the usual moment problem.

V.4 - Proposition : Let $\mu = \sum_{i=1}^{\ell} \mu_i \delta_{\sigma_i}$; and

$$s_{2k} = \int_{\mathbb{R}_+} a^{2k} d\mu(a) \quad k \text{ an integer. Then}$$

$$(12) \quad \begin{vmatrix} s_{2k+2,0} & s_{2k+2} & \cdots & s_{2k+2\ell} \\ s_{2k+2,1} & s_{2k+4} & \cdots & s_{2k+2\ell+2} \\ \cdots & \cdots & \cdots & \cdots \\ s_{2k+2\ell} & s_{2k+2\ell} & \cdots & s_{2k+4\ell} \end{vmatrix} = 0$$

This determinant has $\ell + 1$ rows.

Proof : The proof is written for the case of $\ell = 2$. We start from

$$\mu_1 + \mu_2 = 1$$

$$\mu_1 \sigma_1^2 + \mu_2 \sigma_2^2 = s_2$$

$$\mu_1 \sigma_1^4 + \mu_2 \sigma_2^4 = s_4$$

$$\mu_1 \sigma_1^6 + \mu_2 \sigma_2^6 = s_6$$

These equations are linear in μ_1, μ_2 , the first two determine μ_1, μ_2 and the other two must be consistent.

$$\begin{vmatrix} 1 & 1 & 1 \\ \sigma_1^2 & \sigma_2^2 & s_2 \\ \sigma_1^4 & \sigma_2^4 & s_4 \end{vmatrix} = 0 \text{ and}$$

$$\begin{vmatrix} \sigma_1^2 & \sigma_2^2 & s_2 \\ \sigma_1^4 & \sigma_2^4 & s_4 \\ \sigma_1^6 & \sigma_2^6 & s_6 \end{vmatrix} = \sigma_1^2 \sigma_2^2 \begin{vmatrix} 1 & 1 & s_2 \\ \sigma_1^2 & \sigma_2^2 & s_4 \\ \sigma_1^4 & \sigma_2^4 & s_6 \end{vmatrix} = 0$$

$$\text{Let } v_1 \triangleq \begin{vmatrix} 1 \\ \sigma_1^2 \\ \sigma_1^4 \end{vmatrix} \quad v_2 \triangleq \begin{vmatrix} 1 \\ \sigma_2^2 \\ \sigma_2^4 \end{vmatrix} \text{ and } s_0 \triangleq \begin{vmatrix} 1 \\ s_2 \\ s_4 \end{vmatrix} \quad s_1 \triangleq \begin{vmatrix} s_2 \\ s_4 \\ s_6 \end{vmatrix}$$

S_0 and S_1 both belong to the span of V_1 and V_2 and any other vector in this space is dependent on S_0 and S_1 . This is the case of

$$S_2 \triangleq \begin{vmatrix} S_4 \\ S_6 \\ S_8 \end{vmatrix} \quad \text{since the relation}$$

$$\mu_1 \sigma_1^8 + \mu_2 \sigma_2^8 = S_8 \quad \text{yields}$$

$$\begin{vmatrix} \sigma_1^4 & \sigma_2^4 & S_4 \\ \sigma_1^6 & \sigma_2^6 & S_6 \\ \sigma_1^8 & \sigma_2^8 & S_8 \end{vmatrix} = \sigma_1^4 \sigma_2^4 \begin{vmatrix} 1 & 1 & S_4 \\ \sigma_1^2 & \sigma_2^2 & S_6 \\ \sigma_1^4 & \sigma_2^4 & S_8 \end{vmatrix} = 0$$

So finally

$$\begin{vmatrix} 1 & S_2 & S_4 \\ S_2 & S_4 & S_6 \\ S_4 & S_6 & S_8 \end{vmatrix} = 0$$

Starting from the equations

$$\begin{aligned} \mu_1 \sigma_1^{2k} + \mu_2 \sigma_2^{2k} &= S_{2k} \\ \mu_1 \sigma_1^{2k+2} + \mu_2 \sigma_2^{2k+2} &= S_{2k+2} \\ \dots &\dots \dots \\ \mu_1 \sigma_1^{2k+8} + \mu_2 \sigma_2^{2k+8} &= S_{2k+8} \end{aligned}$$

A similar result is obtained, that is

$$\begin{vmatrix} S_{2k} & S_{2k+2} & S_{2k+4} \\ S_{2k+2} & S_{2k+4} & S_{2k+6} \\ S_{2k+4} & S_{2k+6} & S_{2k+8} \end{vmatrix} = 0$$

The generalization for arbitrary k is obvious. ■

We next show that λ which appears in (11) is the solution of a linear equation.

Indeed by dividing (11), by λ , yields

$$\mu_1 + \mu_2 + \dots + \mu_\ell = 1$$

$$\mu_1 \sigma_1^2 + \mu_2 \sigma_2^2 + \dots + \mu_\ell \sigma_\ell^2 = \frac{\alpha_1}{\lambda} \triangleq s_2$$

.....

$$\mu_1 \sigma_1^{4\ell} + \mu_2 \sigma_2^{4\ell} + \dots + \mu_\ell \sigma_\ell^{4\ell} = \frac{\alpha_{2\ell}}{\lambda} \triangleq s_{4\ell}$$

and by proposition (V.4), one has

$$\begin{vmatrix} 1 & \frac{\alpha_1}{\lambda} & \dots & \frac{\alpha_\ell}{\lambda} \\ \frac{\alpha_1}{\lambda} & \frac{\alpha_2}{\lambda} & \dots & \frac{\alpha_{\ell+1}}{\lambda} \\ \frac{\alpha_\ell}{\lambda} & \frac{\alpha_{\ell+1}}{\lambda} & \dots & \frac{\alpha_{2\ell}}{\lambda} \end{vmatrix} = 0 \text{ or}$$

$$\begin{vmatrix} \lambda & \alpha_1 & \dots & \alpha_\ell \\ \alpha_1 & \alpha_2 & \dots & \alpha_{\ell+1} \\ \alpha_\ell & \alpha_{\ell+1} & \dots & \alpha_{2\ell} \end{vmatrix} = 0 \text{ which can be written as a linear equation as follows :}$$

$$(13) \quad \lambda \begin{vmatrix} \alpha_2 & \dots & \alpha_{\ell+1} \\ \dots & \dots & \dots \\ \alpha_{\ell+1} & \dots & \alpha_{2\ell} \end{vmatrix} = - \begin{vmatrix} 0 & \alpha_1 & \dots & \alpha_\ell \\ \alpha_1 & \alpha_2 & \dots & \alpha_{\ell+1} \\ \dots & \dots & \dots & \dots \\ \alpha_\ell & \alpha_{\ell+1} & \dots & \alpha_{2\ell} \end{vmatrix}$$

Remarks : If μ is as above, there are relations between the α_p 's which must be satisfied because of proposition V.4

For example, one must have :

$$(14) \quad \begin{vmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{\ell+1} \\ \alpha_{\ell+1} & \alpha_{\ell+2} & \dots & \alpha_{2\ell+1} \end{vmatrix} = 0$$

If these relations are not satisfied where calculations with experimental data are carried out, one must chose a larger value of l .

The gaussian case is associated with the value $l = 1$ and the equality $\alpha_1 \alpha_3 - \alpha_2^2 = \int a^2 d\mu \int a^4 d\mu - \left| \int a^3 d\mu \right|^2 = 0$. Thus one must compute moments of order six.

When $\hat{\mu}$ is a convex combination of point masses, the Hilbert space $L^2(\mathbb{R}_+, \mathbb{R}_+, \hat{\mu})$ can be characterized, indeed

V.5 - Proposition : If $\hat{\mu}$ is a convex mixture of l dirac, the Hilbert space $L^2(\mathbb{R}_+, \mathbb{R}_+, \hat{\mu})$ has dimension $2l$.

Proof : By corollary V.2, one has an explicit expression for P_{2l} , which is orthogonal to $1, a, \dots, a^{2l-1}$.

$$P_{2l}(a) = \frac{1}{\sqrt{D_{2l} D_{2l-1}}} \begin{vmatrix} 1 & s_2 & \dots & s_{2l} \\ s_2 & s_4 & \dots & s_{2l+1} \\ \dots & \dots & \dots & \dots \\ 1 & a^2 & \dots & a^{2l} \end{vmatrix}$$

$$\text{Moreover } \int P_{2l}(a) a^{2l} d\hat{\mu}(a) = \begin{vmatrix} 1 & s_2 & \dots & s_{2l} \\ s_2 & s_4 & \dots & s_{2l+2} \\ \dots & \dots & \dots & \dots \\ s_{2l} & s_{2l+2} & \dots & s_{4l} \end{vmatrix} = 0$$

by proposition V.4. With respect to the measure μ , one has $\int |P_{2l}(a)|^2 d\mu(a) = 0$ and P_{2l} is linearly dependent of $1, a, \dots, a^{2l-1}$.

The same occurs for P_{2l+1} and more generally $\forall k > 0$ for P_{2l+k} . ■

Thus $P_{2\ell}$ is a linear combination of $1, P_\ell, \dots, P_{2\ell-1}$ but $P_{2\ell}$ is useful for the identification :

V.6 - Proposition : The roots of $P_{2\ell}$ are precisely

$$\pm \sigma_1 \quad \pm \sigma_2, \dots, \pm \sigma_\ell$$

Proof : The proof is again written for the case of $\ell = 2$; one has :

$$P_4(a) = \begin{vmatrix} 1 & s_2 & s_4 \\ s_2 & s_4 & s_6 \\ 1 & a^2 & a^4 \end{vmatrix}$$

$$\text{But } P_4(\sigma_1) = \begin{vmatrix} \mu_1 + \mu_2 & \mu_1 \sigma_1^2 + \mu_2 \sigma_2^2 & \mu_1 \sigma_1^4 + \mu_2 \sigma_2^4 \\ \mu_1 \sigma_1^2 + \mu_2 \sigma_2^2 & \mu_1 \sigma_1^4 + \mu_2 \sigma_2^4 & \mu_1 \sigma_1^6 + \mu_2 \sigma_2^6 \\ 1 & \sigma_1^2 & \sigma_1^4 \end{vmatrix} \quad \text{so that}$$

$$P_4(\sigma_1) = \begin{vmatrix} \mu_2 & \mu_2 \sigma_2^2 & \mu_2 \sigma_2^4 \\ \mu_2 \sigma_2^2 & \mu_2 \sigma_2^4 & \mu_2 \sigma_2^6 \\ 1 & \sigma_1^2 & \sigma_1^4 \end{vmatrix} = 0$$

The general case is similar. ■

Once the polynomials $P_0, P_1, \dots, P_{2\ell-1}$ are known the values of μ_k follow $k = 1, \dots, \ell$ since ([1] p. 22).

$$\frac{1}{2} \mu_k = \frac{1}{\sum_{i=0}^{2\ell-1} |P_i(\sigma_k)|^2}$$

An other solution to compute μ_k $k = 1, 1, \dots, \ell$ would be to consider the ℓ first equations of system (10), (11) since λ and $\sigma_1, \dots, \sigma_\ell$ are yet computed.

In conclusion, when it is assumed that C has a spherically invariant law rather than a Gaussian law and when the mixing term μ is a convex combination of point masses

$$\mu \triangleq \sum_{k=1}^{\ell} \mu_k \delta_{\sigma_k},$$

we can adjust the moments of C with the experimental moments of C till order $4\ell-1$, and using the extra parameter λ of the Poisson process equalities hold till order 4ℓ . The extra moments of order $4\ell+k$ are determined by the shape of μ (see proposition V.4) but there is no more moments to check discrepancies between the chosen model and the experimental data.

BIBLIOGRAPHIE

- [1] N. I. AKHIEZER, The Classical Moment Problem
University Mathematical Monographs, Oliver & Boyd,
London, 1965.
- [2] D. DUGUE, Traité de Statistique théorique et appliquée,
Masson et Cie, Paris, 1958.
- [3] M. DYM, H. P. KEAN, Gaussian Processes, Function Theory
and the Inverse Spectral Problem, Academic Press, N. Y.,
San Francisco, London, 1976.
- [4] P. A. MEYER, Probabilités et Potentiel, Herman, 1966.
- [5] B. PICINBONO, Spherically invariant and compound gaussian
stochastic processes, IEEE Transactions on information
theory, vol. I, T. 6, 1970, pages 77-79.
- [6] G. RUCKEBUSH, Sur le problème de la Synthèse des filtres,
Colloque C. N. R. S., Aussois, 21-25 sept. 1981
Volume 2, p. A1-1, A1.21
I. N. R.I. A., BP 105, 78153, Le Chesnay, France.

